Stability analysis of mean-field-type nonlinear Fokker-Planck equations associated with a generalized entropy and its application to the self-gravitating system

Masatoshi Shiino

Department of Applied Physics, Faculty of Science, Tokyo Institute of Technology, 2-12-1 Ohokayama, Meguro-ku, Tokyo, Japan
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Multidimensional nonlinear Fokker-Planck equations of mean-field type are proposed within the framework of generalized thermostatistics to develop a general formulation of stability analysis of their solutions. Two types of eigenvalue equations are studied. The nonlinear Fokker-Planck equations are shown to exhibit an *H* theorem with a Liapunov functional that takes the form of a free energy involving generalized entropies of Tsallis. The second-order variation of the Liapunov functional is computed to conduct local stability analysis and the associated eigenvalue equation is derived for an arbitrary form of mean-field coupling potential. Assuming quasiequilibrium for the velocity distribution, the reduced eigenvalue equation with space coordinates alone is also obtained. The alternative type of eigenvalue equation based on the linearization of the nonlinear Fokker-Planck equations is presented. Taking the mean-field coupling potential to be the gravitational one, the nonlinear Fokker-Planck equation in terms of three-dimensional velocity and space coordinates together with the framework of stability analysis is shown to be applicable to a mean-field model of self-gravitating system. By solving the eigenvalue equation for the eigenfunction with 0 eigenvalue, the occurrence of stability change of the equilibrium probability density with spherical symmetry is discussed.

I. INTRODUCTION

Fokker-Planck equations (FPEs) are very useful for studying dynamical as well as equilibrium statistical behaviors of stochastic systems and physical systems in contact with heat reservoirs. They usually take the form of a linear master equation governing the time evolution of the probability density of a system with Markovian dynamics. The approach to the uniquely determined equilibrium or stationary probability density is ensured by an *H* theorem [1–6]. The *H* functional or Liapunov functional taking the form of Kullback-Leibler divergence [8] or relative entropy is closely related to the concept of free energy [6,7]. Such a situation involving an *H* theorem makes the FPE a smart mathematical model allowing for a satisfactory stochastic description of irreversible processes of physical systems.

Recently, Fokker-Planck equations that are different from the standard type are becoming the subject of an intense research [9–35]. Among them are nonlinear Fokker-Planck equations (NFPEs), which are classified into the ones associated with nonlinear diffusion [36–40] and those of mean-field type.

The NFPEs with a nonlinear diffusion term, which have been proposed in connection with generalized thermostatistics developed by Tsallis [41–43], are closely related to the generalized nonextensive entropies [41–43]. One of such NFPEs reads [25]

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}p\right) + D\frac{\partial^2}{\partial x^2}p^q. \quad (1.1)$$

Its equilibrium solution takes the form of Tsallis equilibrium distribution of the first choice [41,43],

$$P_{eq}(x) = \{(Dq\beta)^{-1}[1-\beta(q-1)\phi(x)]\}^{1/(q-1)}, \quad (1.2)$$

which is given by applying the maximum entropy principle for the generalized entropy [41–43],

$$S \equiv \frac{1}{q-1}\left[1 - \int p^q dx\right]. \quad (1.3)$$

The relationship between such NFPE (we will call it the NFPE with a generalized entropy) and the generalized entropy has been studied in detail by several authors [25,30,35]. It has been found that the NFPE with a generalized entropy proposed as an extension of the standard FPE bears close resemblance to the latter with respect to the convergence and stability properties involving an *H* theorem [30,35]: an *H* theorem holds true to ensure uniqueness and stability of its equilibrium solution and the Liapunov functional takes the form of free energy based on the generalized entropy. In the case of Eq. (1.1), for example, the Liapunov functional is given by [30]

$$F = U - DS = \int \phi p\, dx - \frac{D}{q-1}\left[1 - \int p^q dx\right] \quad (1.4)$$

and according to the *H* theorem the *F* monotonically decreases with time to approach its equilibrium value

$$F_{eq} = -D\int p_{eq}^q(x)dx + \frac{1}{q-1}\left(\frac{1}{\beta} - D\right). \quad (1.5)$$

Furthermore, the thermodynamic relations arising from the Legendre transform structure hold regarding the equilibrium free energy F_{eq} [31]:

$$\frac{\partial F_{eq}(D,h)}{\partial D} = -S, \quad (1.6)$$

$$\frac{\partial F_{\text{eq}}(D,h)}{\partial h} = -\langle x \rangle_{P_{\text{eq}}} \equiv - \int x p_{\text{eq}}(x;D,h)dx, \quad (1.7)$$

where $F_{\text{eq}}(D,h)$ is defined for the equilibrium distribution (1.2) of NFPE (1.1) with ϕ replaced by $\phi - hx$. It should be noted that Eqs. (1.6) and (1.7) together with Eq. (1.4) will imply that the first choice of generalized thermostatistics of Tsallis makes sense with D playing the role of temperature [30]. In other words, it is due to the thermodynamic relations (1.6) and (1.7) as well as the dynamic level definition of F of Eq. (1.4) yielding an H theorem that the F becomes the properly defined free energy associated with entropy (1.3). Then the corresponding NFPE (1.1), which is specific to the entropy, makes sense in that the maximal entropy principle for determining an equilibrium distribution can be extended naturally to a dynamic level prescription for obtaining it based on the corresponding generalized canonical ensemble consistent with the thermodynamic stability.

In contrast to the above mentioned type of NFPE, which is free from the occurrence of bifurcations in spite of its nonlinearity, another type of NFPEs that are based on the nonlinearity arising from mean-field-type feedback effect can exhibit bifurcation phenomena. The simplest one of such NFPEs is given by [11,12,14,15]

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\left(x - x^3 + \varepsilon \int xp\, dx\right)p\right] + D \frac{\partial^2}{\partial x^2}p, \quad (1.8)$$

which is derived in the thermodynamic limit for a mean-field coupled Langevin equation system. Such a mean-field type of NFPE is quite convenient to observe the effect of noise on a variety of cooperative phenomena of coupled systems [9–23] and to systematically study equilibrium as well as nonequilibrium phase transitions [18–23].

The problem of phase transitions within the framework of Tsallis thermostatistics is considered to be worth studying. I have recently proposed a double nonlinear Fokker-Planck equation (DNFPE) [31,32] that is obtained by introducing such a mean-field-type nonlinear term as considered in Eq. (1.8) into the NFPE (1.1),

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\left(-\frac{\partial \phi}{\partial x} + \varepsilon \int xp\, dx\right)p\right] + D \frac{\partial^2}{\partial x^2}p^q, \quad (1.9)$$

to study bifurcation phenomena within the context of convergence to equilibrium solutions involving their global and local stability.

It was found that when $\phi(x)$ is chosen to be a double well potential $\phi(x) = -(x^2/2) + (x^4/4)$, the system characterized by a power-law-type equilibrium probability density (1.2) exhibits a pitch-fork bifurcation in a similar manner to the case of the standard mean-field model based on Eq. (1.8), as the control parameter D is varied. Stability analysis was conducted on the basis of calculating the second-order variation of the free energy functional serving as a Liapunov functional [31].

It will be of interest to extend the above mentioned DNFPE to study behaviors of a wide class of DNFPE with a more general type of mean-field coupling [32] than the fer-

romagnetic coupling. Furthermore, it will also be of value to attempt to find physical systems to which the NFPE or DNFPEs can be applied to cope with difficulties encountered with use of the standard Boltzmann-Gibbs entropy.

The aim of this paper is twofold: one is to study multidimensional NFPEs of mean-field type associated with generalized entropies, for the purpose of developing stability analysis for the case with a general type of mean-field potential rather than the ferromagnetic coupling used in Eq. (1.9). Second, I want to apply the results obtained to study a mean-field model of a self-gravitating system [44–53].

The approach based on Tsallis thermostatistics to the problem of a self-gravitating system was first made by Plastino and Plastino [48]. They applied Tsallis's formalism to the problem of stellar polytropes, which was first studied by Kelvin [49] and dealt with in detail by Chandrasekher [50], and discussed the relationship between the index q and the stellar polytrope index. They noted that the standard treatment based on the Boltzmann-Gibbs entropy, in which maximizing the entropy is performed with the constraints imposed by conservation of mass and energy, breaks down with divergence of mass.

Quite recently, stability analysis required for the maximum entropy principle has been conducted within the framework of Tsallis thermostatistics by Taruya and Sakagami [52]. They reported the occurrence of instability for polytrope index n larger than 5.

Imposing constraints of mass and energy for maximizing a certain entropy corresponds to taking the microcanonical ensemble approach to the problem of a self-gravitating system, where some spatial confinement of particles is usually required. Then a problem arises of what will happen to the case of taking the canonical ensemble approach. In this case, one deals with the free energy rather than the entropy itself by viewing the parameter playing the role of temperature as a fixed control parameter. When taking advantage of Tsallis equilibrium distribution (1.2) exhibiting high energy cutoff with $q > 1$, it will become possible to investigate the problem by employing the minimum free energy principle under the constraint of mass alone without considering any spatial confinement.

To be more precise, the mean-field model for a self-gravitating system, which will turn out to constitute the problem of stellar polytropes in the microcanonical and canonical ensemble approaches, will be formulated within the context of generalized canonical ensemble approach as follows.

Consider the energy [48]

$$E_{\text{tot}} = \frac{1}{2} \int \vec{u}^2 p(\vec{u},\vec{x})d^3\vec{u}\, d^3\vec{x} + \frac{1}{2} \int \Gamma(\vec{x})p(\vec{u},\vec{x})d^3\vec{u}\, d^3\vec{x} \quad (1.10)$$

with

$$\Gamma(\vec{x}) = -k \int \frac{p(\vec{u},\vec{z})}{|\vec{x}-\vec{z}|}d^3\vec{u}\, d^3\vec{z}, \quad (1.11)$$

where $p(\vec{u},\vec{x})$ denotes a probability density and k the gravitational constant. Taking the total mass to be unity, the problem is the variational one to minimize the free energy $F = E_{tot} - DS$ under conservation of probability.

In the present study we take the entropy S to be given by Eq. (1.3) for simplicity.

We show that an appropriately taken multidimensional DNFPE with the gravitational potential for the mean-field coupling turns out to work in consistence with undertaking the study based on the above mentioned approach. The free energy, which is claimed to decrease with time in accordance with H theorem, is to be locally minimized for an equilibrium density to be relevant (stable).

The paper is organized as follows. In Sec. II, we present the simplest model of multidimensional DNFPEs of mean-field type within the context of generalized thermostatistics. Linearizing the DNFPE around its equilibrium solution we obtain an eigenvalue equation to discuss the structure of the 0-eigenvalue function(s). An H theorem is shown to hold by taking the free energy as the Liapunov functional and its second-order variation is obtained. In Sec. III, the stability analysis based on the second-order variation of the Liapunov functional is developed further for a more realistic case where the NFPE has physical variables denoting space coordinates and velocity in the three-dimensional physical space. Eigenvalue equations corresponding to the second-order variation of the Liapunov functional are derived. In Sec. IV, we deal with the second-order variation by restricting a perturbation of the probability density to the subspace including the eigenfunction with 0 eigenvalue, which corresponds to considering quasiequilibrium for the velocity distribution. We present the reduced eigenvalue equation with space coordinates alone. In Sec. V, taking the mean-field coupling potential to be the gravitational one we show that the nonlinear Fokker-Planck equation together with the framework of stability analysis in Sec. III can be applied to a mean-field model of self-gravitating system. We solve the eigenvalue equation to obtain its eigenfunction with 0 eigenvalue for the purpose of investigating the occurrence of stability change of the equilibrium probability density with spherical symmetry. In Sec. VI, we give a summary and discussion.

II. MULTIDIMENSIONAL DOUBLE NONLINEAR FOKKER-PLANCK EQUATION AND H THEOREM

As the simplest case of multidimensional systems, we first consider a double nonlinear Fokker-Planck equation of the form

$$\frac{\partial p(t,x,y)}{\partial t} = -\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}p(t,x,y)\right) + D\frac{\partial^2}{\partial x^2}p(t,x,y)^q$$

$$-\frac{\partial}{\partial y}\left[\left(-\frac{\partial \phi}{\partial y}-\int\int\frac{\partial}{\partial y}V(y,z)p(t,x,z)dx\,dz\right)p(t,x,y)\right] + D\frac{\partial^2}{\partial y^2}p(t,x,y)^q, \qquad (2.1)$$

where $\phi(x,y)$ represents a general potential function, $V(y,z)$ another potential corresponding to a mean-field type coupling, and D a positive constant.

Defining

$$\Omega(x,y,p(t,\cdot))\equiv\phi(x,y)+\int\int V(y,z)p(t,x,z)dx\,dz, \qquad (2.2)$$

$$J_1[p]=-\frac{\partial\Omega}{\partial x}p-D\frac{\partial}{\partial x}p^q, \qquad (2.3)$$

$$J_2[p]=-\frac{\partial\Omega}{\partial y}p-D\frac{\partial}{\partial y}p^q, \qquad (2.4)$$

we can rewrite Eq. (2.1) as

$$\frac{\partial p}{\partial t}=-\left(\frac{\partial J_1[p]}{\partial x}+\frac{\partial J_2[p]}{\partial y}\right), \qquad (2.5)$$

which implies conservation of probability under the natural boundary condition or an analogous one that ensures vanishing of the probability current on the boundary. A function p_{eq} that satisfies

$$\Omega(x,y,p_{eq}(\cdot))=\frac{Dq\beta p_{eq}(x,y)^{q-1}-1}{(1-q)\beta} \qquad (2.6)$$

with β denoting some constant, turns out to be an equilibrium solution to Eq. (2.1) because it yields

$$J_1[p_{eq}]=J_2[p_{eq}]=0. \qquad (2.7)$$

We can formally solve for p_{eq},

$$p_{eq}(x)=\{(Dq\beta)^{-1}[1-\beta(q-1)\Omega(x,y,p_{eq}(\cdot))]\}^{1/(q-1)}, \qquad (2.8)$$

where β is determined by normalization condition. Since Ω contains p_{eq}, uniqueness and stability of the p_{eq}, in general, cannot be expected [31,32].

Linear stability analysis

The approach of $p(t,\vec{x},\vec{y})$ to the equilibrium solution can be examined by the linear stability analysis. We linearize the DNFPE (2.1) around the equilibrium solution (2.8) by putting $\delta p=p-p_{eq}$. Noting Eqs. (2.3), (2.4), and (2.6) we obtain

$$\frac{\partial \delta p}{\partial t} = -\frac{\partial}{\partial x}\left[\left(\frac{Dq\frac{\partial}{\partial x}p_{eq}^{q-1}}{q-1}\right)\delta p\right] + Dq\frac{\partial^2}{\partial x^2}p_{eq}^{q-1}\delta p$$

$$-\frac{\partial}{\partial y}\left[\left(\frac{Dq\frac{\partial}{\partial y}p_{eq}^{q-1}}{q-1}\right)\delta p\right] + Dq\frac{\partial^2}{\partial y^2}p_{eq}^{q-1}\delta p$$

$$+\frac{\partial}{\partial y}\left[p_{eq}\int\frac{\partial}{\partial y}V(y,z)\delta p(t,x,z)dx\,dz\right]. \qquad (2.9)$$

Rewriting this equation and putting $\delta p(t,x,y) = \text{const}\times\exp(-\lambda t)f(x,y)$, we obtain the eigenvalue equation

$$-\lambda f(x,y) = Dqp_{eq}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[p_{eq}^{q-2}f]$$

$$+p_{eq}\frac{\partial^2}{\partial y^2}\int V(y,z)f(x,z)dx\,dz$$

$$+Dq\left(\frac{\partial p_{eq}}{\partial x}\frac{\partial}{\partial x}(p_{eq}^{q-2}f) + \frac{\partial p_{eq}}{\partial y}\frac{\partial}{\partial y}(p_{eq}^{q-2}f)\right)$$

$$+\frac{\partial p_{eq}}{\partial y}\frac{\partial}{\partial y}\int V(y,z)f(x,z)dx\,dz. \qquad (2.10)$$

Stability of the equilibrium solution is ensured by the condition that all of the eigenvalues λ be positive. The eigenvalue $\lambda = 0$ is of particular interest, since the occurrence of stability change of the solutions is inferred from the 0 crossing of the smallest eigenvalue as a control parameter like D is changed. The eigenvalue equation (2.10), however, has always 0 eigenvalue irrespective of values of the parameter, as can be seen below.

The equilibrium densities (2.8) containing the integration constant β constitute a family of fixed point solutions of the DNFPE, unless normalization is considered. This implies that eigenvalue equation (2.10) always yields 0 eigenvalue. Indeed, according to this observation we can search for the eigenfunction(s) with 0 eigenvalue. Taking the differential of Eq. (2.6) with respect to a change in β, we obtain

$$Dq\ p_{eq}^{q-2}\delta p = -\int\int V(y,z)\delta p(x,z)dx\,dz - \frac{1}{1-q}\delta\left(\frac{1}{\beta}\right). \qquad (2.11)$$

The δp satisfying this equation should be the eigenfunction with $\lambda = 0$. Multiplying both hands of Eq. (2.11) by the gradient operator, we obtain

$$\text{grad}\left(Dq\ p_{eq}^{q-2}\delta p + \int\int V(y,z)\delta p(x,z)dx\,dz\right) = \vec{0}. \qquad (2.12)$$

It is easy to see that this equation implies Eq. (2.10) with $\lambda = 0$ for $\delta p = f$. We can recover Eq. (2.12) conducting another type of stability analysis based on the second-order variation of the H functional in Sec. III.

We note that the δp given by Eq. (2.11) does not always satisfy the condition

$$\int\int\delta p(x,y)dx\,dy = 0, \qquad (2.13)$$

which is necessary for probability conservation. Accordingly, relevant eigenfunctions with $\lambda = 0$, which is related to the occurrence of bifurcations, have to be given by Eq. (2.12) satisfying the condition (2.13).

H theorem

The behavior of the approach of $p(t,\vec{x},\vec{y})$ to the equilibrium solution can also be examined with the help of an H theorem as in the case of one-dimensional DNFPEs [14,15,31].

We define the free energy functional as

$$F(p(\cdot)) = U - DS \qquad (2.14)$$

where $p(x,y)$ denotes a probability density and

$$U = \int\int\phi(x,y)p(x,y)dx\,dy + \frac{1}{2}\int\int V(y,z)p(x_1,y)p(x_2,z)dx_1dx_2dy\,dz$$

$$= \int\int\left[\phi(x,y) + \int\int V(y,z)p(x,z)dx\,dz\right]p(x,y)dx\,dy - \frac{1}{2}\int\int V(y,z)p(x_1,y)p(x_2,z)dx_1dx_2dy\,dz, \qquad (2.15)$$

$$S = \frac{1}{q-1}\left[1 - \int\int p^q dx\,dy\right]. \qquad (2.16)$$

Substituting the solution of Eq. (2.1) into Eq. (2.14) [i.e., $p = p(t,x,y)$], we differentiate F with respect to t. Using Eqs. (2.1) and (2.2) we have

$$\frac{dF(p(t,\cdot))}{dt}=\int\int\left[\Omega(x,y,p(t,\cdot))+\frac{Dq}{q-1}p(t,x,y)^{q-1}\right]\frac{\partial p(t,x,y)}{\partial t}dx\,dy$$

$$=\int\int\left[\Omega+\frac{Dq}{q-1}p^{q-1}\right]\left[-\frac{\partial}{\partial x}\left(-\frac{\partial\Omega}{\partial x}p-D\frac{\partial}{\partial x}p^q\right)-\frac{\partial}{\partial y}\left(-\frac{\partial\Omega}{\partial y}p-D\frac{\partial}{\partial y}p^q\right)\right]dx\,dy. \tag{2.17}$$

We define $R_p(t,x,y)$ together with β so as to satisfy the following equations:

$$\Omega(x,y,p(t,\cdot))=\frac{Dq\beta R_p(t,x,y)^{q-1}-1}{(1-q)\beta}, \tag{2.18}$$

$$\int\int R_p(t,x,y)dx\,dy=1. \tag{2.19}$$

The unique existence of such β can be confirmed if the $\Omega(x,y,p(t,\cdot))$ as a function of x and y is bounded from below [30,31].

Performing integration by parts, we rewrite Eq. (2.17):

$$\frac{dF(p(t,\cdot))}{dt}=-\int\int\left[\frac{\partial}{\partial x}\left(\Omega+\frac{Dq}{q-1}p^{q-1}\right)\right]\left(\frac{\partial\Omega}{\partial x}p+D\frac{\partial}{\partial x}p^q\right)dx\,dy$$

$$-\int\int\left[\frac{\partial}{\partial y}\left(\Omega+\frac{Dq}{q-1}p^{q-1}\right)\right]\left(\frac{\partial\Omega}{\partial y}p+D\frac{\partial}{\partial y}p^q\right)dx\,dy$$

$$=-\left(\frac{Dq}{q-1}\right)^2\int\int p\left[\left\{\frac{\partial}{\partial x}(p^{q-1}-R_p^{q-1})\right\}^2+\left\{\frac{\partial}{\partial y}(p^{q-1}-R_p^{q-1})\right\}^2\right]dx\,dy. \tag{2.20}$$

Accordingly, we have

$$\frac{dF(p(t,\cdot))}{dt}\leq0, \tag{2.21}$$

where equality sign holds under the condition

$$p^{q-1}-R_p^{q-1}=\text{const.} \tag{2.22}$$

We note here that Eqs. (2.20) and (2.21) hold true even in the case with $q>1$, where the domain for the integration involved may happen to be subjected to the so-called high energy cutoff to become time dependent. This is because we may safely assume that on the domain boundary the probability currents (2.3) and (2.4) vanish.

If the free energy F is bounded from below, inequality (2.21) implies that for large times $p(t,x,y)$ approaches R_p, because noting Eq. (2.19) the above condition holds only for $p=R_p$.

Hence the equilibrium probability density $p_{\text{eq}}(x,y)$ must be determined from the self-consistent equation (2.6) as described earlier.

It is noted that one can no longer expect uniqueness of the equilibrium density $p_{\text{eq}}(x,y)$, since the above self-consistent equation, in general, admits multisolutions. Which of those multisolutions is relevant and is approached from an appropriately given initial condition has to be determined by the stability condition. In other words, there may occur bifurcation phenomena involving stability switches, when control parameters such as D are changed.

To conduct local stability analysis [14,15,31], we expand F around p_{eq},

$$\delta F\equiv F(p_{\text{eq}}+\delta p)-F(p_{\text{eq}})=\delta^{(1)}F[\delta p]+\delta^{(2)}F[\delta p,\delta p]$$

$$+\cdots. \tag{2.23}$$

Noting Eq. (2.13) and differentiating F with respect to p one has

$$\delta F=\frac{Dq}{q-1}\int\int(p^{q-1}-R_p^{q-1})\delta p\,dx\,dy. \tag{2.24}$$

Differentiating F twice yields

$$2\delta^{(2)}F=Dq\int\int(p^{q-2}\delta p-R_p^{q-2}\delta R_p)\delta p\,dx\,dy. \tag{2.25}$$

Since δR_p satisfies

$$\int\int V(y,z)\delta p(x,z)dx\,dz=-DqR_p^{q-2}\delta R_p$$

$$-\frac{1}{1-q}\frac{\delta}{\delta p}\left(\frac{1}{\beta}\right), \tag{2.26}$$

which is obtained by differentiating Eq. (2.18), we have

$$2\,\delta^{(2)}F = Dq \int \int p^{q-2}(\delta p)^2\, dx\, dy$$

$$+ \int \int \int \int V(y,z)\,\delta p(x,y)\,\delta p(x',z)\,dy\, dz\, dx\, dx'.$$

(2.27)

Putting $p = p_{\mathrm{eq}}$ in Eqs. (2.24) and (2.27), one obtains

$$\delta^{(1)}F[\delta p] = \frac{Dq}{q-1} \int \int (p_{\mathrm{eq}}^{q-1} - R_p^{q-1})\,\delta p\, dx\, dy = 0,$$

(2.28)

$$2\,\delta^{(2)}F[\delta p,\ \delta p]$$

$$= Dq \int \int p_{\mathrm{eq}}^{q-2}(\delta p)^2\, dx\, dy$$

$$+ \int \int \int \int V(y,z)\,\delta p(x,y)\,\delta p(x',z)\,dy\, dz\, dx\, dx'.$$

(2.29)

where V represents an arbitrary function that is chosen depending on models one considers and the multiple integral is represented by an abbreviated expression of using the single integral. The ϕ may express another potential giving rise to an external force or some others. An interesting case, however, will be the one where we take the potential ϕ to be the kinetic energy $\phi(\vec{u},\vec{x}) = \frac{1}{2}\vec{u}^2$. In this case the system governed by Eq. (3.1) will turn out to correspond to a mean-field model with generalized thermostatistics for particles interacting via the interaction potential V, which will be studied by taking V to be the gravitational potential in the later section. In what follows in this section, however, we do not assume any particular form for ϕ and V. We see that the variables x and y in Eq. (2.1) of the preceding section are replaced with $\vec{u}$ and $\vec{x}$, respectively, in the above DNFPE.

As in the case of the DNFPE (2.1), we obtain formally an equilibrium density

$$p_{\mathrm{eq}}(\vec{u},\vec{x}) = \{(Dq\beta)^{-1}[1 - \beta(q-1)\Omega(\vec{u},\vec{x},p_{\mathrm{eq}}(\cdot))]\}^{1/(q-1)}$$

(3.2)

with

$$\Omega(\vec{u},\vec{x},p(t,\cdot)) = \phi(\vec{u},\vec{x}) + \int V(\vec{x},\vec{z})p(t,\vec{u},\vec{z})\,d\vec{u}\, d\vec{z}$$

(3.3)

We can also derive the eigenvalue equation associated with the second-order variation (2.29), which will be studied in the following section.

III. STABILITY ANALYSIS AND EIGENVALUE EQUATION

To further develop stability analysis based on the second-order variation (2.29) of the H functional for a more realistic situation, we deal with a higher-dimensional case of the NFPE (2.1) that has physical variables $\vec{x}$ and $\vec{u}$ denoting, respectively, the space coordinate and velocity in the three-dimensional physical space. We consider a certain interaction potential V that is supposed to act between particles.

Denoting by $p(t,\vec{u},\vec{z})$ the probability density to find a particle at a state with space coordinate $\vec{x}$ and velocity $\vec{u}$ in the six-dimensional configurational space at time t, we consider the time evolution of the probability density to be given by the following NFPE:

$$\frac{\partial p(t,\vec{u},\vec{x})}{\partial t} = \sum_{k=1}^{3} \left\{ -\frac{\partial}{\partial u_k}\left(-\frac{\partial\phi}{\partial u_k}p(t,\vec{u},\vec{x})\right) + D\frac{\partial^2}{\partial u_k^2}p(t,\vec{u},\vec{x})^q \right\}$$

$$+ \sum_{k=1}^{3} \left\{ -\frac{\partial}{\partial x_k}\left[\left(-\frac{\partial\phi}{\partial x_k} - \int \frac{\partial}{\partial x_k}V(\vec{x},\vec{z})p(t,\vec{u},\vec{z})\,d\vec{u}\, d\vec{z}\right)p(t,\vec{u},\vec{x})\right] + D\frac{\partial^2}{\partial x_k^2}p(t,\vec{u},\vec{x})^q \right\},$$

(3.1)

With the free energy functional constructed in a similar way as in Eq. (2.14), the H theorem of the preceding section holds also for the NFPE (3.1). Then, the second-order variation reads

$$2\,\delta^{(2)}F[\delta p,\ \delta p]$$

$$= \int G(\vec{u},\vec{x},\vec{w},\vec{z})\,\delta p(\vec{u},\vec{x})\,\delta p(\vec{w},\vec{z})\,d\vec{u}\, d\vec{w}\, d\vec{x}\, d\vec{z}$$

(3.4)

with

$$G(\vec{u},\vec{x},\vec{w},\vec{z}) = V(\vec{x},\vec{z}) + Dq\ p_{\mathrm{eq}}^{q-2}(\vec{u},\vec{x})\,\delta(\vec{w}-\vec{u})\,\delta(\vec{z}-\vec{x}).$$

(3.5)

We proceed further to develop the local stability analysis of the NFPE (3.1) on the basis of the above second-order variation of the free energy functional.

Considering the condition for δp

$$\int \delta p(\vec{u},\vec{x})\,d\vec{u}\, d\vec{x} = 0,$$

(3.6)

we put

$$\delta p(\vec{u},\vec{x}) = \sum_{i=1}^{3} \frac{\partial A_i(\vec{u},\vec{x})}{\partial u_i} + \sum_{i=1}^{3} \frac{\partial A_{3+i}(\vec{u},\vec{x})}{\partial x_i} \equiv \operatorname{div} \vec{A}(\vec{u},\vec{x}), \tag{3.7}$$

where $A_i(\vec{u},\vec{x})$ $(i=1,\ldots,6)$ denote arbitrary functions that vanish for $|\vec{u}| \to \infty$ and $|\vec{x}| \to \infty$ or on the boundary of the domain one is considering.

Then performing integration by parts we have

$$\int V(\vec{x},\vec{z}) \delta p(\vec{u},\vec{x}) \delta p(\vec{w},\vec{z}) d\vec{u}\, d\vec{w}\, d\vec{x}\, d\vec{z} = \int V(\vec{x},\vec{z}) \left(\sum_{i=1}^{3} \frac{\partial A_{3+i}(\vec{u},\vec{x})}{\partial x_i} \right) \left(\sum_{j=1}^{3} \frac{\partial A_{3+j}(\vec{w},\vec{z})}{\partial z_j} \right) d\vec{u}\, d\vec{w}\, d\vec{x}\, d\vec{z}$$

$$= \sum_{ij=1}^{3} \int \frac{\partial^2 V(\vec{x},\vec{z})}{\partial x_i \partial z_j} A_{3+i}(\vec{u},\vec{x}) A_{3+j}(\vec{w},\vec{z}) d\vec{u}\, d\vec{w}\, d\vec{x}\, d\vec{z} \tag{3.8}$$

and also

$$\int Dq\; p_{\text{eq}}^{q-2}(\vec{u},\vec{x}) \delta(\vec{w}-\vec{u}) \delta(\vec{z}-\vec{x}) \delta p(\vec{u},\vec{x}) \delta p(\vec{w},\vec{z}) d\vec{u}\, d\vec{w}\, d\vec{x}\, d\vec{z} = \sum_{ij=1}^{3} Dq \int (I_{ij}^{(1)} + I_{ij}^{(2)} + I_{ij}^{(3)} + I_{ij}^{(4)}) d\vec{u}\, d\vec{w}\, d\vec{x}\, d\vec{z} \tag{3.9}$$

with

$$I_{ij}^{(1)} = \left[\frac{\partial^2}{\partial u_i \partial w_j} [p_{\text{eq}}^{q-2}(\vec{u},\vec{x}) \delta(\vec{w}-\vec{u}) \delta(\vec{z}-\vec{x})] \right] A_i(\vec{u},\vec{x}) A_j(\vec{w},\vec{z}),$$

$$I_{ij}^{(2)} = \left[\frac{\partial^2}{\partial u_i \partial z_j} [p_{\text{eq}}^{q-2}(\vec{u},\vec{x}) \delta(\vec{w}-\vec{u}) \delta(\vec{z}-\vec{x})] \right] A_i(\vec{u},\vec{x}) A_{3+j}(\vec{w},\vec{z}),$$

$$I_{ij}^{(3)} = \left[\frac{\partial^2}{\partial x_i \partial w_j} [p_{\text{eq}}^{q-2}(\vec{u},\vec{x}) \delta(\vec{w}-\vec{u}) \delta(\vec{z}-\vec{x})] \right] A_{3+i}(\vec{u},\vec{x}) A_j(\vec{w},\vec{z}),$$

$$I_{ij}^{(4)} = \left[\frac{\partial^2}{\partial x_i \partial z_j} [p_{\text{eq}}^{q-2}(\vec{u},\vec{x}) \delta(\vec{w}-\vec{u}) \delta(\vec{z}-\vec{x})] \right] A_{3+i}(\vec{u},\vec{x}) A_{3+j}(\vec{w},\vec{z}). \tag{3.10}$$

Defining T_{kl} $(1 \leq k,l \leq 6)$ as

$$T_{kl} = Dq\, \frac{\partial^2}{\partial u_k \partial w_l} [p_{\text{eq}}^{q-2}(\vec{u},\vec{x}) \delta(\vec{w}-\vec{u}) \delta(\vec{z}-\vec{x})], \quad 1 \leq k,l \leq 3,$$

$$T_{kl} = Dq\, \frac{\partial^2}{\partial u_k \partial z_{l-3}} [p_{\text{eq}}^{q-2}(\vec{u},\vec{x}) \delta(\vec{w}-\vec{u}) \delta(\vec{z}-\vec{x})], \quad 1 \leq k \leq 3,\ 4 \leq l \leq 6,$$

$$T_{kl} = Dq\, \frac{\partial^2}{\partial x_{k-3} \partial w_l} [p_{\text{eq}}^{q-2}(\vec{u},\vec{x}) \delta(\vec{w}-\vec{u}) \delta(\vec{z}-\vec{x})], \quad 4 \leq k \leq 6,\ 1 \leq l \leq 3,$$

$$T_{kl} = \frac{\partial^2}{\partial x_{k-3} \partial z_{l-3}} [Dq\; p_{\text{eq}}^{q-2}(\vec{u},\vec{x}) \delta(\vec{w}-\vec{u}) \delta(\vec{z}-\vec{x}) + V(\vec{x},\vec{z})], \quad 4 \leq k,l \leq 6, \tag{3.11}$$

we have

$$2 \delta^{(2)} F[\delta p, \delta p] = \int \sum_{k=1}^{6} \sum_{l=1}^{6} T_{kl}(\vec{u},\vec{x},\vec{w},\vec{z}) A_k(\vec{u},\vec{x}) A_l(\vec{w},\vec{z}) d\vec{u}\, d\vec{w}\, d\vec{x}\, d\vec{z}. \tag{3.12}$$

Accordingly the eigenvalue equation associated with the above quadratic form takes the form

$$\sum_{l=1}^{6} \int T_{kl}(\vec{u},\vec{x},\vec{w},\vec{z})A_l(\vec{w},\vec{z})d\vec{w}\,d\vec{z}=\lambda A_k(\vec{u},\vec{x}), \quad k=1,\ldots,6. \tag{3.13}$$

By performing integration by parts, the above equation can be rewritten as follows:

$$-\frac{\partial}{\partial u_k}[Dq\,p_{\mathrm{eq}}^{q-2}(\vec{u},\vec{x})\mathrm{div}\,\vec{A}(\vec{u},\vec{x})]=\lambda A_k(\vec{u},\vec{x}), \quad k=1,\ldots,3 \tag{3.14}$$

$$-\frac{\partial}{\partial x_{k-3}}\left[Dq\,p_{\mathrm{eq}}^{q-2}(\vec{u},\vec{x})\mathrm{div}\,\vec{A}(\vec{u},\vec{x})+\int V(\vec{x},\vec{z})\sum_{l=4}^{6}\frac{\partial A_l(\vec{w},\vec{z})}{\partial z_{l-3}}d\vec{w}\,d\vec{z}\right]=\lambda A_k(\vec{u},\vec{x}), \quad k=4,\ldots,6. \tag{3.15}$$

By multiplying both hands of Eq. (3.14) and Eq. (3.15) by $\partial/\partial u_k$ and $\partial/\partial x_{k-3}$, respectively, and summing them up we obtain

$$-\Delta_{(\vec{u},\vec{x})}[Dq\,p_{\mathrm{eq}}^{q-2}(\vec{u},\vec{x})\mathrm{div}\,\vec{A}(\vec{u},\vec{x})]-\Delta_{\vec{x}}\int V(\vec{x},\vec{z})\sum_{l=4}^{6}\frac{\partial A_l(\vec{w},\vec{z})}{\partial z_{l-3}}d\vec{w}\,d\vec{z}=\lambda\,\mathrm{div}\,\vec{A}(\vec{u},\vec{x}), \tag{3.16}$$

where

$$\Delta_{(\vec{u},\vec{x})}\equiv\sum_{k=1}^{3}\frac{\partial^2}{\partial u_k^2}+\sum_{k=1}^{3}\frac{\partial^2}{\partial x_k^2}, \quad \Delta_{\vec{x}}\equiv\sum_{k=1}^{3}\frac{\partial^2}{\partial x_k^2}. \tag{3.17}$$

Putting

$$\mathrm{div}\,\vec{A}(\vec{u},\vec{x})\equiv\gamma(\vec{u},\vec{x}) \tag{3.18}$$

and noting

$$\int \mathrm{div}\,\vec{A}(\vec{u},\vec{x})d\vec{u}=\int\sum_{k=1}^{3}\frac{\partial A_{k+3}(\vec{u},\vec{x})}{\partial x_k}d\vec{u}, \tag{3.19}$$

Eq. (3.16) can be rewritten as

$$-\Delta_{(\vec{u},\vec{x})}[Dq\,p_{\mathrm{eq}}^{q-2}(\vec{u},\vec{x})\gamma(\vec{u},\vec{x})]$$

$$-\Delta_{\vec{x}}\int V(\vec{x},\vec{z})\gamma(\vec{w},\vec{z})d\vec{w}\,d\vec{z}=\lambda\,\gamma(\vec{u},\vec{x}), \tag{3.20}$$

which should be solved under the condition

$$\int \gamma(\vec{u},\vec{x})d\vec{u}\,d\vec{x}=0 \tag{3.21}$$

to find the eigenvalues.

It is noted that Eq. (3.20) must be supplemented by another condition that is required to ensure Eqs. (3.14) and (3.15) on the basis of Eq. (3.20), because we have taken the divergence operation in deriving Eq. (3.16). It will read

$$\mathrm{grad}_{(\vec{u},\vec{x})}[Dq\,p_{\mathrm{eq}}^{q-2}(\vec{u},\vec{x})\gamma(\vec{u},\vec{x})]$$

$$+\mathrm{grad}_{\vec{x}}\int V(\vec{x},\vec{z})\gamma(\vec{w},\vec{z})d\vec{w}\,d\vec{z}\to 0 \tag{3.22}$$

when $\vec{u},\vec{x}$ approach the domain boundary where $\vec{A}(\vec{u},\vec{x})$ vanishes.

As an interesting application of the eigenvalue equation, we can consider the case where V is given by the gravitational potential

$$V(\vec{x},\vec{z})=\frac{-k}{|\vec{x}-\vec{z}|} \tag{3.23}$$

with k denoting a positive constant, for which

$$\Delta_{\vec{x}}\left(\frac{-k}{|\vec{x}-\vec{z}|}\right)=4\pi k\,\delta(\vec{x}-\vec{z}). \tag{3.24}$$

Then Eq. (3.20) takes the form

$$-\Delta_{(\vec{u},\vec{x})}[Dq\,p_{\mathrm{eq}}^{q-2}(\vec{u},\vec{x})\gamma(\vec{u},\vec{x})]-4\pi k\int \gamma(\vec{w},\vec{x})d\vec{w}$$

$$=\lambda\,\gamma(\vec{u},\vec{x}). \tag{3.25}$$

As in the preceding section, we can obtain a similar eigenvalue equation by linearizing the DNFPE (3.1) . It will be straightforward, by noting the counterpart (2.10) for the DNFPE of the preceding section, to write down the eigenvalue equation:

$$-\lambda f(\vec{u},\vec{x}) = Dq\, p_{eq}\Delta_{(\vec{u},\vec{x})}[p_{eq}^{q-2}f] + p_{eq}\Delta_{\vec{x}}\int V(\vec{x},\vec{z})f(\vec{u},\vec{z})d\vec{u}\,d\vec{z} + Dq(\text{grad}_{(\vec{u},\vec{x})}p_{eq})$$

$$\times[\text{grad}_{(\vec{u},\vec{x})}(p_{eq}^{q-2}f)] + (\text{grad}_{\vec{x}}p_{eq})\left(\text{grad}_{\vec{x}}\int V(\vec{x},\vec{z})f(\vec{u},\vec{z})d\vec{u}\,d\vec{z}\right). \quad (3.26)$$

We now have two eigenvalue equations (3.20) and (3.26) for our DNFPE (3.1), which seem to differ from each other. The eigenfunction with 0 eigenvalue, however, should be the same, because the 0 eigenvalue is related to the fixed point solution of the DNFPE as well as the stability switch of the system. We indeed observe that this is the case. When we put $\lambda = 0$ in the eigenvalue equations (3.14) and (3.15), we obtain

$$\text{grad}_{\vec{u}}(p_{eq}^{q-2}\gamma) = 0, \quad (3.27)$$

$$Dq\,\text{grad}_{\vec{x}}(p_{eq}^{q-2}\gamma) + \text{grad}_{\vec{x}}\int V(\vec{x},\vec{z})\gamma(\vec{u},\vec{z})d\vec{u}\,d\vec{z} = 0. \quad (3.28)$$

Accordingly, combining the above equations and the eigenvalue equation (3.20) with $\lambda = 0$,

$$\Delta_{(\vec{u},\vec{x})}[Dq\; p_{eq}^{q-2}(\vec{u},\vec{x})\gamma(\vec{u},\vec{x})] + \Delta_{\vec{x}}\int V(\vec{x},\vec{z})\gamma(\vec{w},\vec{z})d\vec{w}\,d\vec{z}$$

$$= 0, \quad (3.29)$$

which originally has been derived from Eqs. (3.27) and (3.28), we can easily see that Eq. (3.26) with $\lambda = 0$ holds for $\gamma = f$.

The eigenvalue equations (3.20) and (3.26) are six-dimensional integropartial differential equations and hard to solve, since the standard method of separation of variables does not work.

To obtain a more easy-to-solve simplified eigenvalue equation with a reduced number of variables, for example, working with space variables alone it will be necessary to confine ourselves to a certain subspace in which the perturbation δp is considered for the second-order variation (3.4) of the Liapunov functional.

IV. EIGENVALUE EQUATION IN TERMS OF THE SPACE VARIABLES

In view of the fact that the equilibrium density takes the form (3.2), we assume the perturbation δp to be given by

$$p_{eq} + \delta p = [(Dq\tilde{\beta})^{-1}[1 - \tilde{\beta}(q-1)\{\Omega(\vec{u},\vec{x},p_{eq}(\cdot))$$

$$+ \delta\Gamma(\vec{x})\}]]^{1/(q-1)}, \quad (4.1)$$

where $\tilde{\beta}$ denotes the normalization constant and $\delta\Gamma(\vec{x})$ an arbitrary function. Then it follows

$$\delta p(\vec{u},\vec{x}) = -(Dq)^{-1}p_{eq}^{2-q}\left(\delta\Gamma(\vec{x}) - \frac{\int p_{eq}^{2-q}\delta\Gamma(\vec{x})d\vec{u}\,d\vec{x}}{\int p_{eq}^{2-q}d\vec{u}\,d\vec{x}}\right). \quad (4.2)$$

We note that the assumption of Eq. (4.1) may be related to that of quasiequilibrium for the distribution of velocity.

We define the followings:

$$\mu(\vec{x}) \equiv (Dq)^{-1}\int p_{eq}^{2-q}(\vec{u},\vec{x})d\vec{u}, \quad (4.3)$$

$$\mu_0 \equiv \int \mu(\vec{x})d\vec{x}, \quad (4.4)$$

$$\zeta(\vec{x}) \equiv \mu(\vec{x})\delta\Gamma(\vec{x}) - \frac{\mu(\vec{x})}{\mu_0}\int \mu(\vec{x})\delta\Gamma(\vec{x})d\vec{x}. \quad (4.5)$$

Substituting Eq. (4.2) into the second-order variation (3.4), we obtain

$$2\delta^{(2)}F = \int \frac{1}{\mu(\vec{x})}\zeta(\vec{x})^2d\vec{x} + \int V(\vec{x},\vec{z})\zeta(\vec{x})\zeta(\vec{z})d\vec{x}\,d\vec{z}. \quad (4.6)$$

It is noted that

$$\int \zeta(\vec{x})d\vec{x} = 0. \quad (4.7)$$

Repeating a similar line of reasoning in the derivation of Eq. (3.20) for the reduced second-order variation (4.6) with Eq. (4.7), we straightforwardly obtain the following eigenvalue equation:

$$-\Delta\left[\frac{1}{\mu(\vec{x})}\zeta(\vec{x})\right] - \Delta\int V(\vec{x},\vec{z})\zeta(\vec{z})d\vec{z} = \lambda\zeta(\vec{x}). \quad (4.8)$$

The eigenvalue equation with $\lambda = 0$ is of particular concern. Putting $\lambda = 0$ into Eq. (4.8) one obtains

$$-\Delta[\delta\Gamma(\vec{x})] - \Delta\int V(\vec{x},\vec{z})\mu(\vec{z})$$

$$\times\left(\delta\Gamma(\vec{z}) - \frac{1}{\mu_0}\int \mu(\vec{y})\delta\Gamma(\vec{y})d\vec{y}\right)d\vec{z} = 0. \quad (4.9)$$

It is worth noting that the above equation is the same as what is obtained by substituting Eq. (4.2) into the original 0-eigenvalue equation (3.29). This implies that the eigenfunction with 0 eigenvalue of the original eigenvalue equa-

tion (3.29) can be given by the perturbation (4.2) and then it suffices to solve Eq. (4.9). The occurrence of such a situation can also easily be understood by noting Eq. (3.27). When $\lambda = 0$, Eq. (3.27) implies that $p_{eq}^{q-2}\delta p$ is independent of the velocity variables and hence becomes a function of only the space coordinates as given by Eq. (4.2).

V. APPLICATION TO A MEAN-FIELD MODEL OF THE SELF-GRAVITATING SYSTEM

To observe effectiveness of the eigenvalue equation obtained in Sec. IV for somewhat realistic physical model, we deal with a mean-field model of the self-gravitating system whose energy is given by Eq. (1.10) under the canonical ensemble approach and apply the formulation of stability analysis of the previous sections. To this end we consider ϕ to represent the kinetic energy

$$\phi(\vec{u},\vec{x}) = \tfrac{1}{2}\vec{u}^2 \qquad (5.1)$$

and $V(\vec{x},\vec{z})$ the gravitational potential (3.23). Then the energy U in Eq. (2.15) coincides with that of the self-gravitating system: $U = E_{tot}$. Taking S to be the Tsallis entropy of the form (2.16), we now see that finding the local minima of the free energy $F = E_{tot} - DS$ of the self-gravitating system with respect to the density $p(\vec{u},\vec{x})$ corresponds to solving the stable equilibrium solutions to our DN-FPE (3.1), which is associated with that entropy, owing to the H theorem that ensures monotonic decreasing of the F with time. We note that the use of Tsallis entropy makes the system polytropic [48] and enables one to deal with a self-confined state of the system.

We also note that the astrophysical relevance of the DN-FPE (3.1) to the dynamical evolution equation of the self-gravitating particles is beyond the scope of this paper and remains a future problem, because the right hand side of Eq. (3.1) includes the diffusion in position space.

To make the paper self-contained, some of the well known established results of the equilibrium properties of the polytropes are reproduced before presenting the results of application of the stability analysis in the mean-field model, part of which was already reported in Ref. [54]. Confining ourselves only to a self-confined state to assume $q > 1$, we rewrite the equilibrium density (3.2) of the DNFPE as

$$p_{eq}(\vec{u},\vec{x}) = A(q,D)[-\tfrac{1}{2}\vec{u}^2 - \Gamma_{eq}(\vec{x}) + B(q,\beta)]^{1/(q-1)}, \qquad (5.2)$$

where

$$A = \left(\frac{q-1}{Dq}\right)^{1/(q-1)}, \quad B = \frac{1}{(q-1)\beta}, \qquad (5.3)$$

$$\Gamma_{eq}(\vec{x}) = -k\int \frac{p_{eq}(\vec{u},\vec{z})}{|\vec{x}-\vec{z}|}d\vec{u}\,d\vec{z} = -k\int \frac{\rho(\vec{z})}{|\vec{x}-\vec{z}|}d\vec{z} \quad (5.4)$$

with

$$\rho(\vec{x}) = \int p_{eq}(\vec{u},\vec{x})d\vec{u}. \qquad (5.5)$$

Substitution of Eq. (5.2) into Eq. (5.5) gives

$$\rho(\vec{x}) = A\,4\pi C\left(\frac{1}{q-1}\right)[B - \Gamma_{eq}(\vec{x})]^{3/2 + 1/(q-1)} \qquad (5.6)$$

with

$$C(m) = \int_o^{\sqrt{2}} h^2\left(1 - \frac{1}{2}h^2\right)^m dh \qquad (5.7)$$

which can be defined for $m > -1$.

The reason for assuming $q > 1$ comes from the requirement of convergence of $\vec{x}$ integration of $p_{eq}(\vec{u},\vec{x})$: Since it is expected from Eq. (5.4) that $\Gamma_{eq}(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$, assuming $0 < q < 1$ in Eq. (3.2) leads to the divergence of $\vec{x}$ integration of p_{eq}. On the other hand, choosing $q > 1$ can avoid such divergence of the p_{eq} by making the domain of $\vec{x}$ integration for p_{eq} limited to a certain bounded domain of the three-dimensional space for which the p_{eq} is well defined. To be more specific, with $q > 1$ we consider the $p_{eq}(\vec{u},\vec{x})$ to be defined for the domain satisfying

$$-\tfrac{1}{2}\vec{u}^2 - \Gamma_{eq}(\vec{x}) + B \geq 0, \quad B < 0 \quad (\text{i.e.,} \quad \beta < 0), \quad (5.8)$$

for which integration with respect to $d\vec{u}\,d\vec{z}$ is meant to be performed. Then the domain of $\vec{x}$ integration allowed becomes $|\vec{x}| \leq r_c$ with $B - \Gamma_{eq}(|\vec{x}| = r_c) = 0$ if, for simplicity, spherical symmetry of $\Gamma(\vec{x})$ can be assumed. In other words, we can say that the bounded domain where $p_{eq} > 0$ has been naturally introduced according to our prescription based on the high energy cutoff.

Taking the Laplacian of Eq. (5.4) one obtains the Poisson equation

$$\Delta\Gamma_{eq}(\vec{x}) = k\int 4\pi\delta(\vec{x}-\vec{z})\rho(\vec{z})d\vec{z} = 4\pi k\rho(\vec{x}). \quad (5.9)$$

When we can assume that $\Gamma_{eq}(\vec{x})$ and $\rho(\vec{x})$ have spherical symmetry, Eq. (5.9) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\Gamma_{eq}(r)\right) = 4\pi k\rho(r). \qquad (5.10)$$

Substituting Eq. (5.6) one obtains

$$-\left[A\,4\pi C\left(\frac{1}{q-1}\right)\right]^{-\omega}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\rho^\omega\right) = 4\pi k\rho \qquad (5.11)$$

with

$$\omega = \frac{2(q-1)}{3q-1}, \qquad (5.12)$$

which is a well known equation in the polytrope gas system [44,49–52]. The ω is the inverse of the polytrope index n: $\omega = 1/n$ [51,52]. The equilibrium density should be given by solving this second-order differential equation under the boundary conditions, which read

$$\left.\frac{d}{dr}\rho^{\omega}\right|_{r=0}=0, \tag{5.13}$$

$$\left.\frac{d}{dr}\rho^{\omega}\right|_{r=r_c}=-k\left[4\pi AC\left(\frac{1}{q-1}\right)\right]^{\omega}\frac{1}{r_c^2}. \tag{5.14}$$

Equation (5.13) arises from the condition that the solution to Eq. (5.11) must satisfy the hydropressure balance condition for the equilibrium density [44,49–52].

Equation (5.14) follows from the normalization of the probability density. We further impose the condition that our system be subjected to the high energy cutoff at $r=r_c$ based on the choice of the nonextensive index of $q>1$ for Tsallis statistics,

$$\rho(r_c)=0, \tag{5.15}$$

which determines the value of r_c.

It is often more convenient to deal with a scaled nondimensional equation of Eq. (5.11). Putting as

$$\rho(r)=\rho_0\tilde{\rho}(\xi), \quad r=r_0\xi, \tag{5.16}$$

where

$$r_0^2=\frac{1}{4\pi k}\left[A4\pi C\left(\frac{1}{q-1}\right)\right]^{-\omega}\rho_0^{\omega-1}, \tag{5.17}$$

Eq. (5.11) can be rewritten as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d}{d\xi}\tilde{p}^{\omega}\right)=-\tilde{\rho}, \tag{5.18}$$

which should be solved under the boundary condition that

$$\tilde{\rho}(0)=1, \quad \left.\frac{d}{d\xi}\tilde{p}^{\omega}\right|_{\xi=0}=0. \tag{5.19}$$

In general, the ρ_0 as an integration constant can be determined by the normalization condition by assuming that the integration of the original $\rho(r)$ is limited to the interval $[0,\eta]$. Then the condition corresponding to Eq. (5.14) reads

$$r_0^3\rho_0\int_0^{\eta/r_0}4\pi\xi^2\tilde{\rho}(\xi)d\xi=1 \tag{5.20}$$

or equivalently

$$-4\pi r_0\rho_0\eta^2\left.\frac{d}{d\xi}\tilde{p}^{\omega}\right|_{\eta/r_0}=1, \tag{5.21}$$

which gives ρ_0 and r_0 as a function of η. In our case we consider that $\eta=r_c$. Hence the r_c is to be determined from a special value $\xi_c=r_c/r_0$ satisfying

$$\tilde{\rho}(\xi_c)=0. \tag{5.22}$$

A family of solutions $\tilde{\rho}^{\omega}$ satisfying Eq. (5.18) together with the boundary condition (5.19) is called the Emden so-

lution [44,49–52]. It is known that for the polytrope index $n=1/\omega<5$, the solution rapidly decreases to vanish at a finite value of ξ, satisfying Eq. (5.22), whereas for $n=1/\omega>5$ the solution decays slowly only to vanish at $\xi=\infty$.

We now turn to study the eigenvalue problem of this system. To examine whether there occurs a stability change with the appearance of 0 eigenvalue for the equilibrium density satisfying Eq. (5.11), we solve Eqs. (3.27) and (3.28) to find eigenfunctions. Equation (3.27) together with Eq. (3.21) turns out to be satisfied by solutions of the form (4.2).

Substituting the $\delta p(\vec{u},\vec{x})$ in Eq. (4.2) into the eigenvalue equation (3.28), one obtains the equation for $\delta\Gamma(\vec{x})$:

$$\mathrm{grad}\left[\delta\Gamma(\vec{x})+\int V(\vec{x},\vec{z})\mu(\vec{z})\right.$$
$$\left.\times\left(\delta\Gamma(\vec{z})-\frac{1}{\mu_0}\int\mu(\vec{y})\delta\Gamma(\vec{y})d\vec{y}\right)d\vec{z}\right]=0. \tag{5.23}$$

Multiplying Eq. (5.23) by the divergence operator, one obtains

$$\Delta_{\vec{x}}\delta\Gamma(\vec{x})+4\pi k\mu(\vec{x})\left(\delta\Gamma(\vec{x})-\frac{\int\mu(\vec{x})\delta\Gamma(\vec{x})d\vec{x}}{\int\mu(\vec{x})d\vec{x}}\right)=0, \tag{5.24}$$

where $\mu(\vec{x})$ defined by Eq. (4.3) is explicitly given by

$$\mu(\vec{x})=\frac{1}{\omega}\left[4\pi AC\left(\frac{1}{q-1}\right)\right]^{\omega}\rho^{1-\omega}. \tag{5.25}$$

Equation (5.24) coincides with Eq. (4.9).

Here we assume a spherical symmetric solution $\delta\Gamma(\vec{x})=R(r)$. Then we have from Eq. (5.24)

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)R(r)+4\pi k\mu(r)$$
$$\times\left(R(r)-\frac{\int_0^{r_c}\mu(r)R(r)4\pi r^2dr}{\int_0^{r_c}\mu(r)4\pi r^2dr}\right)=0. \tag{5.26}$$

Putting

$$\bar{R}(r)=R(r)-\frac{\int_0^{r_c}\mu(r)R(r)4\pi r^2dr}{\int_0^{r_c}\mu(r)4\pi r^2dr}, \tag{5.27}$$

Eq. (5.26) can be rewritten as

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\bar{R}(r)\right)=-4\pi k\mu(r)r^2\bar{R}(r), \tag{5.28}$$

$$\int_0^{r_c}\mu(r)\bar{R}(r)4\pi r^2dr=0. \tag{5.29}$$

Equation (5.28) is a second-order linear differential equation for $\bar{R}(r)$ and has two linearly independent solutions.

On the other hand, Eq. (5.23) can be rewritten as

$$\frac{d}{dr}\bar{R}(r) = -4\pi k \frac{1}{r^2}\int_0^r \mu(r)\bar{R}(r)r^2 dr, \qquad (5.30)$$

which is the original 0-eigenvalue equation to be solved under the condition (5.29). Differentiating this equation recovers Eq. (5.28). Hence we obtain the boundary condition that Eq. (5.28) has to satisfy:

$$r^2 \frac{d}{dr}\bar{R}(r)\Big|_{r=0} = 0. \qquad (5.31)$$

Accordingly, one has to solve the second-order linear differential equation (5.28) under the condition of Eqs. (5.29) and (5.31) to find 0 eigenvalue.

We can easily find one of the solutions to Eq. (5.28) as

$$\overline{R_1}(r) = \frac{1}{4\pi r^2 \mu(r)}\left[\frac{d}{dr}[4\pi r^3 \rho(r)] + \alpha 4\pi r^2 \rho(r)\right]$$

$$= \omega\left[A 4\pi C\left(\frac{1}{q-1}\right)\right]^{-\omega}\left[(\alpha+3)\rho^\omega + \frac{r}{\omega}\frac{d}{dr}\rho^\omega\right] \qquad (5.32)$$

with

$$\alpha = \frac{1-3\omega}{\omega-1} = \frac{3q-5}{q+1}, \qquad (5.33)$$

which can be confirmed by a direct substitution into Eq. (5.28).

General solutions to Eq. (5.28) can easily be obtained by putting $\bar{R}(r) = \tau(r)\overline{R_1}(r)$. The result is

$$\bar{R}(r) = \overline{R_1}(r)\left(c_1 + c_2 \int_1^r \frac{dr}{r^2 \overline{R_1}^2}\right). \qquad (5.34)$$

Noting Eq. (5.13), we can easily check that boundary condition (5.31) implies $c_2 = 0$. So the $\overline{R_1}(r)$ is the desired solution to the legitimate 0-eigenvalue equation (5.30). Hence it will suffice to consider this solution to see whether the normalization condition (5.29) is satisfied or not. However, it does not satisfy Eq. (5.29) for $\alpha \neq 0$, since noting Eq. (5.15) one has [54]

$$\int_0^{r_c} \mu(r)\overline{R_1}(r)4\pi r^2 dr = \int_0^{r_c} \alpha 4\pi r^2 \rho(r)dr = \alpha, \qquad (5.35)$$

which yields 0 only when $\omega = \frac{1}{3}$. This will imply that marginal stability occurs only at $\alpha = 0$ because of the imposing of the condition (5.15) and that there occurs no critical point corresponding to the stability change for the equilibrium density with a change in such a control parameter as D. We note that the value $\omega = \frac{1}{3}$ is well known in the stability problem of the polytropes [55].

If, instead, one imposes a different boundary condition, say, a rigid boundary condition where the particles are contained in a sphere with radius η to consider the variational problem for the free energy (2.14) or (3.4), one would be able to expect a phase transition to occur. Indeed, the condition (5.29) yields

$$4\pi \eta^3 \rho(\eta,\eta) = -\alpha, \qquad (5.36)$$

where we have specified η dependence of ρ. This implies that when $\alpha > 0$, namely, $\frac{1}{3} < \omega < \frac{2}{3}$, no stability change occurs. On the other hand, when $\alpha < 0$ ($0 < \omega < \frac{1}{3}$), stability change may occur at $\eta = \eta^*$ satisfying Eq. (5.36). A detailed analysis of the normalization condition (5.21) reveals that $|dr_0/d\eta| \to \infty$ as η approaches η^* from above and also that a pair of equilibrium probability densities $\rho_{r_0 > r_0^*}(r,\eta)$ ($\eta^* \leq \eta < \infty$) and $\rho_{r_0 < r_0^*}(r,\eta)$ ($\eta^* \leq \eta \leq r_c$) are allowed to exist for $\eta \geq \eta^*$, where $r_0^* = r_0(\eta^*)$. Stability exchange occurs between two branches at $\eta = \eta^*$. We cannot determine from the marginal stability analysis alone stability of the equilibrium densities. However, for the case with $\alpha > 0$ the equilibrium density can be considered to be stable, because the second-order variation of the free energy functional (4.6) evaluated in the neighborhood of $\eta = 0$ takes positive values (see the Appendix) and no stability change occurs. In the case of $\alpha < 0$, the branch of equilibrium densities $\rho_{r_0 < r_0^*}(r,\eta)$ containing the self-confined one is expected to exhibit instability due to the marginal stability at $\alpha = 0$ for the case with $\eta = r_c$.

Conclusively, the system with $\alpha > 0$ (i.e., $q > \frac{5}{3}$), which exhibits an equilibrium density undergoing the high energy cutoff (i.e., self-confined state), is stable irrespective of values of temperature D [54]. This result recovers the recent ones obtained by means of different methods [56,57].

VI. SUMMARY AND DISCUSSION

We have extended the double nonlinear Fokker-Planck equations (DNFPEs) of the mean-field type, which were proposed previously to study bifurcation phenomena within the framework of Tsallis thermostatistics, to include a multidimensional case with velocity and space coordinates. Taking the mean-field coupling kernel to be of a general form, we have developed stability analysis to obtain the eigenvalue equations based on two different types of approaches. Our DNFPEs have been shown to exhibit an H theorem based on the H functional taking the form of free energy. We have analyzed the second-order variation of the H functional to derive one of the eigenvalue equations. Assuming quasiequilibrium for the velocity distribution, the reduced eigenvalue equation with space coordinates alone is also obtained. Another eigenvalue equation has been obtained by the standard method of linearizing the original DNFPEs around the equilibrium density. As far as the 0 eigenvalue is concerned, we have shown that the eigenvalue equation associated with the second-order variation of the H functional implies the one obtained from linearization of the DNFPEs.

Taking the mean-field coupling kernel and another potential function involved in the DNFPE to be the gravitational

potential and the kinetic energy of particles, respectively, we have applied the DNFPE having the three-dimensional velocity and position space coordinates together with the result of stability analysis to study the mean-field model of the self-gravitating system.

The eigenfunctions with 0 eigenvalue have been found to be exhaustively given by the reduced eigenvalue equation of position variable. We have examined the marginal stability to obtain the condition for the occurrence of stability change of the equilibrium density given as the Emden solution. As far as the equilibrium density undergoing the high energy cutoff is concerned, no stability change occurs with changes of the coefficient of the nonlinear diffusion term D that plays the role of temperature.

Furthermore, the obtained condition determining stability change shows the existence of a critical value of $\omega = \frac{1}{3}$ corresponding to the marginal stability at $\alpha = 0$ such that for $\omega > \frac{1}{3}$ no stability change occurs even when the rigid boundary condition is deliberately imposed in which the particles are contained in a sphere with radius η. When, on the other hand, $\omega < \frac{1}{3}$, at a certain critical value of η marginal stability occurs in accordance with the occurrence of a saddle-node type bifurcation and the equilibrium densities are allowed to exist only for above the critical value of radius. By evaluating the second-order variation of the free energy functional in the small limit of radius η, we have found that the equilibrium density with $\frac{1}{3} < \omega < \frac{2}{3}$ is stable for any η that must be smaller than the automatically introduced cutoff radius based on Tsallis themostatistics. In particular, the self-confined system with $\frac{1}{3} < \omega < \frac{2}{3}$ (i.e., $q > \frac{5}{3}$) is stable irrespective of values of temperature D.

A few remarks worth noting are in order. Stability issues of the DNFPEs that exhibit an H theorem may simply be studied on the basis of the analysis of the second-order variation of the H functional by making full use of the H theorem. Indeed, in the case of the DNFPEs where the mean-field coupling kernel is given by the ferromagnetic coupling, we previously showed that just computing the second-order variation of the H functional suffices to observe the occurrence of stability change as well as to determine stability of an equilibrium density. There it is not necessary to investigate the eigenvalue equation. This situation is in sharp contrast to the present system, where the eigenvalue equation plays an important role to determine the stability of the equilibrium solution. The difference arises from the type of the mean-field coupling kernel V.

In the case of the ferromagnetic coupling kernel $V(y,z) = -Jyz$ or $V(y,z) = (J/2)(y-z)^2$, diagonalizing the second-order variation can easily be performed to extract the relevant part responsible for the determination of stability [15,31,32]. The phase transition point where stability change occurs corresponds to the appearance of 0 eigenvalue. This can be easily checked by using one-dimensional version of Eq. (3.28) [54].

We note here that the 0-eigenvalue equation itself, in general, can be found from the knowledge of the manifold of the stationary probability density without resorting to the eigenvalue problem.

We have employed the DNFPE approach in the study of

stability of the polytropes, which is essentially based on the canonical ensemble within the context of Tsallis thermostatistics. The reason for dealing with the DNFPE is as follows. Such NFPEs as shown by Eq. (1.1) give an equilibrium solution that exhibits the usually known thermodynamic relation of the Legendre form structure as shown by Eqs. (1.6) and (1.7). It is because of such a relation as well as the dynamical level definition (1.4) of the free energy F that the diffusion parameter D may be interpreted as properly defined temperature and the definition of the F turns out to be appropriate in the generalized thermostatistics. Hence several generalizations of classical results of statistical mechanics may become possible to make sense. In this respect, NFPEs and DNFPEs that are derived from the formal extension of FPE from the $q = 1$ case can be compared to the generalization of entropy proposed by Tsallis. Furthermore, the application of the DNFPE to the mean-field model of the self-gravitating system is based on the very fact that stability issues of the DNFPE and the self-gravitating particles of canonical ensemble approach can be equivalently related with each other via the free energy for which the H theorem of the DNFPE holds. The equilibrium phase space distribution function of the self-gravitating system hence coincides with that of the DNFPE and its marginal distribution of position variable develops polytropic nature.

In general, when the equilibrium distribution function of a system is given for studying its statistical behavior, it is often convenient to consider a certain dynamical equation of the probability distribution (a kind of master equation) which converges to that equilibrium distribution for large times, particularly in the case of systems exhibiting bifurcation phenomena. Numerical simulations of the master equation can often be used not for examining the temporal behavior but just simply for observing the equilibrium properties.

When supposing such a situation, it is considered that physical meaning and astrophysical relevance of the NFPE itself does not matter. Studying this sort of thing is beyond the scope of this paper and remains a future problem.

The static problem of the mean-field model of the self-gravitating system might be considered on the basis of the free energy (2.14) alone without introducing the DNFPE, once the parameter D is viewed as temperature. The advantage of considering the DNFPE will, however, be that the principle of minimizing the free energy can be understood dynamically as the stability issue of the convergence of the equilibrium probability density for large times, although the microscopic foundation of the DNFPE remains an open problem [61]. More specifically, one can easily solve the marginal stability (0-eigenvalue problem) by only dealing with the manifold corresponding to the fixed point solution of the NFPE. The eigenvalue equation obtained by linearization of the DNFPE will also make sense. Furthermore, from the viewpoint of numerical simulations, the equilibrium probability density subjected to the high energy cutoff can be naturally considered as appearing from an initial probability density that has in general no such high energy cutoff.

We have taken advantage of incorporating the cutoff radius corresponding to the high energy cutoff for the equilibrium probability density with $q > 1$ by employing Tsallis

thermostatistics of first choice based on the nonextensive generalized entropy. The r_c has been shown to be given within the equilibrium theory, which does not require the introducing of a rigid boundary condition that is usually taken. To avoid the infinite mass problem that would occur under the use of Boltzmann entropy, we may employ other forms of entropy that bring about a power-law-type equilibrium density responsible for the high energy cutoff. There exist many such entropies, each of which determines a corresponding NFPE such that the free energy involving that entropy decreases with time (H theorem) [54]. For example, we can take a NFPE [30] that is obtained from the Sharma and Mittal entropy [58], instead of the Plastino-Plasino-type NFPE [25], to consider a DNFPE leading to a different free energy for the self-gravitating system. This corresponds to the third choice of Tsallis thermostatistics [43]. In this sense, generalized thermostatistics cannot be determined uniquely.

Regarding stability of the equilibrium probability density of the Emden-type solution (5.16) of the self-gravitating system, its global stability will not be ensured, although the H theorem given by Eq. (2.1) holds. This is because the lower boundedness of the H functional is not shown. The free energy may decrease without bound in order for the system to settle into the collapsed state, where the probability is concentrated to one (center) point.

Comparing our result of stability analysis with the one obtained previously in the microcanonical ensemble approach [52], we note that the condition for stability in terms of the free energy minimum is more stringent than that based on the maximum entropy recipe with the energy constraint. This is because the local minimizing of the free energy implies the local maximizing of the entropy under the constraint of the energy. The present result shows that the critical value of $\omega = \frac{1}{3}$ (inverse of the polytrope index) above which stability follows without exhibiting stability change is larger than that of $\omega = \frac{1}{5}$ for the corresponding phase for the microcanonical ensemble approach [52]. This supports the above argument, which implies that the stability region inferred from the free energy condition should be contained by that from the entropy condition, if a phase diagram in terms of ω and η is drawn.

After completing the present work the author came to know some recent works by other researchers on the related subject of polytropes [56,57,59,60]. Chavanis [56,59], and Taruya and Sakagami [57] studied the canonical ensemble of the polytropes independently of our work to obtain the same eigenvalue equation as the one in the present paper as far as the marginal stability of the eigenvalue equation in terms of the position variables is concerned. The result that the polytropic gas with $\omega > \frac{1}{3}$ exhibits stability is in agreement with the results of Chavanis [56,59] and of Taruya and Sakagami [57]. Chavanis's method in Ref. [56] is different from ours and is based on the dynamical equations involving Euler equation together with the polytropic equation of state. He discussed the difference between the marginal value of $\omega = \frac{1}{3}$ obtained dynamically and that of $\omega = \frac{1}{5}$ obtained by Taruya and Sakagami [52] in the microcanonical ensemble, and emphasized the need for studying the canonical ensemble using a properly chosen free energy to compare them.

This has been indeed done in Refs. [59,57]. Furthermore, Chavanis noted, on the basis of the inspection of his own result in view of Tsallis entropy, that in spite of the difference between the distribution of velocities of the polytropic gas and the equilibrium phase space distribution implied by Tsallis entropy, some connection between the dynamical stability and generalized thermodynamic stability may be expected. The result of the present paper might partially answer this question: The marginal stability of the polytropes can be determined by the reduced eigenvalue equation of the position variable, where the information on the distribution of velocities is masked.

In Ref. [60] Chavanis, Rosier, and Sire studied the thermodynamics and the collapse of a self-gravitating gas of Brownian particles in the high-friction limit to make a direct relation between the dynamics and the thermodynamics from the viewpoint of the standard statistical mechanics based on the Boltzmann-Gibbs entropy. They dealt with the Smoluchowski equation combined with the Poisson equation (SP system) and showed that the SP system satisfies a form of H theorem and the eigenvalue problem for linear stability of a stationary distribution is connected to the eigenvalue problem for the second-order variations of the thermodynamic potentials taken as H functions. Their case corresponds to $q=1$ of our system. Such a system, however, requires confinement of gas particles, where the interesting problem of the gravitational collapse can be expected, to avoid the infinite mass problem and will differ much from the model of the present work dealing with the self-confined case.

In our approach the canonical ensemble is systematically given by elucidating the function of the parameter D playing the role of temperature, which is implied by the thermodynamic relation of the Legendre transform structure and the H theorem for the DNFPE. Furthermore, the present approach taken in deriving the eigenvalue equations from the second-order variation of the free energy to investigate the stability issue of the polytropes is more systematic in that the full phase space variables are taken into account and the use of the reduced eigenvalue equation in terms of position variables can be justified for the issue of the marginal stability.

The detailed analysis of the stability issue of the Emden solution including the phase diagram and the comparisons with other works will be presented elsewhere.

APPENDIX

To observe the behavior of the second-order variation of the free energy functional (4.6) in the neighborhood of $\eta = 0$, we first compute Eq. (4.6) by assuming that the variation $\zeta(\vec{x})$ has the rotational symmetry. One can easily obtain

$$2\delta^{(2)}F = \int_0^\eta \frac{1}{\mu(r,\eta)} \zeta(r)^2 \, 4\pi r^2 dr - k \int_0^\eta dr 4\pi r^2 \frac{\zeta(r)}{r}$$
$$\times \left(\int_0^r 4\pi s^2 \zeta(s) ds + r \int_r^\eta 4\pi s \zeta(s) ds \right). \quad \text{(A1)}$$

Considering Eq. (4.7) we put

$$4\pi r^2 \zeta(r) = \frac{\partial F(r,\eta)}{\partial r}, \qquad (A2)$$

where we have explicitly expressed the η dependence of the F such that

$$F(0,\eta) = F(\eta,\eta) = 0. \qquad (A3)$$

Then it is straightforward to obtain

$$2\delta^{(2)}F = \int_0^\eta \left[\frac{1}{4\pi r^2 \mu(r,\eta)} \left(\frac{\partial F(r,\eta)}{\partial r} \right)^2 - \frac{k}{r^2} F(r,\eta)^2 \right] dr. \qquad (A4)$$

We note here that in order for the above integral to be convergent in the small r, one can assume that

$$\frac{\partial F}{\partial r} \approx r^\sigma, \quad \sigma > \tfrac{1}{2}. \qquad (A5)$$

Let r and η be small and F be analytic for simplicity. Noting Eq. (A3), we expand F as

$$F(r,\eta) = br^2(r-\eta) + \cdots, \qquad (A6)$$

where b is some constant. Using the scaled Emden solution (5.19) and Eq. (5.25) to rewrite $\mu(r,\eta)$, we have

$$2\delta^{(2)}F = \omega \left[A4\pi C \left(\frac{1}{q-1} \right) \right]^{-\omega} \rho_0^{\omega-1} \int_0^\eta \tilde{\rho}\left(\frac{r}{r_0} \right)^{\omega-1} \frac{1}{4\pi r^2}$$

$$\times \left(\frac{\partial F(r,\eta)}{\partial r} \right)^2 dr - k \int_0^\eta \left[\frac{1}{r^2} F(r,\eta)^2 \right] dr. \qquad (A7)$$

When $\alpha > 0$ (i.e., $\tfrac{1}{3} < \omega < \tfrac{2}{3}$), it follows from the normalization condition for the equilibrium density (5.20) that in the limit $\eta \to 0$

$$r_0 \to 0, \quad \frac{\eta}{r_0} \to 0, \quad 4\pi \eta^3 \rho(\eta,\eta) = 4\pi \eta^3 \rho_0 \tilde{\rho}\left(\frac{\eta}{r_0} \right) \to 3. \qquad (A8)$$

Noting the above and substituting Eq. (A6) into Eq. (A7) we obtain in the limit $\eta \to 0$

$$2\delta^{(2)}F \approx \omega \left[A4\pi C \left(\frac{1}{q-1} \right) \right]^{-\omega} \left(\frac{3}{4\pi} \right)^{\omega-1} \frac{b^2}{4\pi} \eta^{-3\omega+6}$$

$$- \frac{1}{30} kb^2 \eta^5. \qquad (A9)$$

Noting $\tfrac{1}{3} < \omega < \tfrac{2}{3}$, we have in the leading order of η positive sign of $\delta^{(2)}F$:

$$\delta^{(2)}F \approx \omega \left[A4\pi C \left(\frac{1}{q-1} \right) \right]^{-\omega} \left(\frac{3}{4\pi} \right)^{\omega-1} \frac{b^2}{4\pi} \eta^{-3\omega+6} > 0. \qquad (A10)$$

Finally, we note that in the case of $\alpha < 0$ one cannot take the limit $\eta \to 0$ in Eq. (5.20), which implies that the equilibrium density based on the Emden solution does not exist for very small η in the case of imposing the rigid boundary condition.

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